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MINIMUM-COST FLOWS IN NETWORKS WITH UPPER BOUNDED ARCS AND CONCAVE COST FUNCTIONS

Wayne Jay Hallenbeck, Jr.

Naval Postgraduate School
Monterey, California

December 1972

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13. ABSTRACT

An algorithm is presented for solving minimum-cost flow problems in which each arc of the network has a finite maximum flow capacity and a concave cost function associated with sending flow along that arc. Each cost function is broken into a series of cost increments through the use of piecewise linear approximations. The algorithm takes any feasible solution and recirculates flow over less costly cycles to obtain an optimal solution. A modification which handles the existence of non-zero lower bounds on flow through the various arcs is also given.

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Minimum-Cost Flows in Networks
with
Upper Bounded Arcs and Concave Cost Functions

by

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TABLE OF SYMBOLS AND ABBREVIATIONS

$G(N,A)$	a connected original network consisting of a set of nodes N and a set of arcs A
(i,j)	an arc directed from node i to node j
M_{ij}	the upper bound on the flow capacity of arc (i,j) of the original network
L_{ij}	the lower bound on the flow capacity of arc (i,j) of the original network
X_{ij}	the actual amount (an integer value) of flow in arc (i,j) of the original network
$C_{ij}(X_{ij})$	the cost of sending X_{ij} units of flow along arc (i,j) of the original network
$C_{ij}^{(k)}$	the incremental cost of sending a kth unit of flow along arc (i,j) of the original network, given that k-1 units have already been sent
Q	the amount of flow to be sent through the original network
Z	the total cost of sending Q units of flow through the original network
$\bar{X}$	a flow vector having all of the various X_{ij} values as its components
$G_\ell(N,A_\ell)$	a connected length sequence network consisting of a set of nodes N and a set of arcs A_ℓ
$\ell_{ij}^{(k)}$	the kth element in the length sequence of arc (i,j) of the length sequence network
d_{ij}	the length of the directed arc from node i to node j in a shortest route algorithm
$V_i^{(k)}$	the label value of node i upon completion of the kth iteration of Dijkstra's algorithm
V_i	the permanent label value of node i

I. INTRODUCTION

A. PAST WORK

Considerable literature exists dealing with minimum-cost network problems in which the various arcs of a network have upper and lower bounds on flow capacity and linear cost functions. In particular, there is the Primal-Dual Algorithm, developed by Ford and Fulkerson [2] in 1955, which solves such network flow problems having all lower bounds equal to zero. Ford and Fulkerson [5] later developed the Out-of-Kilter Algorithm to solve problems having non-zero lower bounds.

The two above algorithms can also be used to deal with minimum-cost network problems in which the cost functions are convex. By making piecewise linear approximations to the convex cost curves, every arc in such a network can be replaced by a group of arcs having different linear cost functions, where each arc of the group corresponds to one segment of the linear approximation to the convex cost function.

Hu [6] has looked at a special case of using piecewise linear approximations in convex-cost networks. He defines "up" and "down" arc costs as the incremental costs incurred from increasing or decreasing, respectively, by one unit of flow, the already existing flow in an arc. He presents a solution procedure which optimally increments, from zero to any desired amount, the total flow through the network. Although Hu does not consider upper bounds on arc flow

capacity, his algorithm can easily be modified to handle this condition by making the "up-cost" of a saturated arc infinite.

However, in many practical applications the convex cost assumption does not hold. Dantzig [1] points out that the presence of a set-up or red tape charge yields a concave cost function. Likewise, it can be argued that efficiencies of scale and the practice of giving discounts or rebates in transactions involving large quantities of goods or services also yield concave cost functions. In general, it can be stated that concave cost functions arise through the existence of decreasing marginal costs, a common phenomenon in real life situations. Thus, there is good reason to study concave cost networks.

Zangwill [8] presents a solution to the minimum-cost flow problem in which the various arcs of a network have concave cost functions, although he does not consider the existence of upper or non-zero lower bounds on the arc capacities. The solution to his problem has all of the flow being sent along the minimum total cost chain of arcs from the source to the sink.

B. OBJECTIVE AND SCOPE

The purpose of this paper is to present a method of solving minimum-cost flow problems in which the arcs of the network have zero lower bounds, finite upper bounds, and concave cost functions.

For any given amount of flow which must be sent from the source to the sink, the problem can be solved if the maximum flow capacity and the cost function are known for each arc in the network. A basic assumption to the solution procedure is that each cost function can be broken into a series of non-increasing cost increments through the use of piecewise linear approximations.

As outputs to the problem, the solution procedure provides the optimal arc flows and the associated optimal total cost.

II. MODEL DEVELOPMENT

A. NETWORK DESCRIPTION AND ASSUMPTIONS

Consider a connected network $G(N,A)$ consisting of a set of nodes N and a set of arcs A. Let the integers $i=1,2,\ldots,n$ represent the nodes and the two-tuples (i,j) $(i=1,2,\ldots,n;\ j=1,2,\ldots,n;$ and $i \neq j)$ represent the arcs. Let node 1 correspond to the source and node n correspond to the sink. The arcs are assumed to be directed so that the order (i,j) implies an arc directed from node i to node j.

It is assumed that there is only one source node and only one sink node. It is also assumed that no more than one arc connects any pair of nodes in the same ordered direction. If multiple sources or sinks exist, artificial nodes and arcs may be added to the network so that there is a single overall source and a single overall sink. Similarly, if two or more directed arcs connect the same pair of nodes in the same ordered direction, artificial nodes and arcs may be added to the network as needed to alleviate this condition.

Associated with each arc are an upper bound on capacity and a cost function for flow over the arc. Let X_{ij} be the actual amount of flow and let M_{ij} be the maximum flow capacity for an arc (i,j). Then, for X_{ij} to be a feasible arc flow,

$$0 \leq X_{ij} \leq M_{ij} \tag{1}$$

for each arc (i,j) in the network.

Conservation of flow is assumed to exist at all nodes. The total flow out of any node equals the total flow into that node; that is, there is no storage of flow at any of the nodes.

B. FORMULATION OF THE COST FUNCTION

Let the concave cost function $C_{ij}(X_{ij})$ represent the cost of sending an amount, X_{ij}, of flow along arc (i,j). It is assumed that $C_{ij}(X_{ij})$ is continuous and non-negative over the entire range (from 0 to M_{ij}) of X_{ij}. It is also assumed that $C_{ij}(0) = 0$ for each arc in the network. However, if there is a cost function such that $C_{ij}(0) \neq 0$, then a new cost function $C'_{ij}(X_{ij}) = C_{ij}(X_{ij}) - C_{ij}(0)$, can be substituted for $C_{ij}(X_{ij})$ without changing the nature of the optimization problem.

It is assumed that piecewise linear approximations can be made to the cost function of every arc in the network. If the arc flows and arc flow capacities are required to be non-negative integers, the cost function of an arc (i,j) can be replaced with a non-increasing sequence of incremental costs: $C_{ij}^{(1)}, C_{ij}^{(2)}, \ldots, C_{ij}^{(M_{ij})}$, where

$$C_{ij}^{(k)} = C_{ij}(k) - C_{ij}(k-1) \tag{2}$$

for $k = 1,2,\ldots,M_{ij}$.

The interpretation here is that an incremental cost, $C_{ij}^{(k)}$, is the additional cost of sending one unit of flow across arc (i,j), given that k-1 units of flow have already been sent across arc (i,j). Thus, if X_{ij} represents the actual amount of flow sent across an arc (i,j), it can easily be seen that the associated cost,

$$C_{ij}(X_{ij}) = C_{ij}^{(0)} + \ldots + C_{ij}^{(X_{ij})}, \tag{3}$$

where $C_{ij}^{(0)}$ is defined to be equal to zero.

C. STATEMENT OF THE COST MINIMIZATION PROBLEM

All flow through the network is assumed to travel from the source node to the sink node. Then, for any specified amount of flow Q to be sent through the network, the cost minimization problem can be stated as the following programming problem:

Find non-negative integer values for all X_{ij} which

$$\text{minimize } Z = \sum_{(i,j) \in A} \left(\sum_{k=0}^{X_{ij}} C_{ij}^{(k)} \right), \tag{4}$$

$$\text{subject to } \sum_j X_{ij} - \sum_h X_{hi} = \begin{cases} Q, & i=1 \\ 0, & i=2,3,\ldots,n-1 \\ -Q, & i=n \end{cases} \tag{5}$$

$$\text{and } \quad 0 \leq X_{ij} \leq M_{ij}, \qquad \forall (i,j) \in A. \tag{6}$$

D. LENGTH SEQUENCE NETWORKS

Let $\bar{X}$ denote a flow vector which has as its components the X_{ij} values of all the arcs in the network. Any flow vector $\bar{X}$ which satisfies constraints (5) and (6) of the

cost minimization problem is a feasible solution to the problem. The set of all flow vectors which are feasible solutions comprise the problem's feasible region.

Consider any flow vector $\bar{X}$ which is a feasible solution to the problem. Associated with this solution, a unique connected length sequence network $G_{\ell}(N,A_{\ell})$ can be constructed. This new network has the same nodes as the original network, but has a different arrangement of arcs, as described below:

1.) If arc (i,j) of the original network is empty, draw a forward arc from node i to node j in the length sequence network. Label this forward arc with the length sequence

$$\ell_{ij}^{(1)}, \ldots, \ell_{ij}^{(M_{ij})}, \text{ where } \ell_{ij}^{(k)} = C_{ij}^{(k)} .$$

2.) If arc (i,j) of the original network is neither empty nor saturated, draw a forward arc from node i to node j and a backward arc from node j to node i in the length sequence network. Label the forward arc with the length

sequence $\ell_{ij}^{(1)}, \ldots, \ell_{ij}^{(M_{ij}-X_{ij})}$, where $\ell_{ij}^{(k)} = C_{ij}^{(X_{ij}+k)}$. Label the backward arc with the length

sequence $\ell_{ji}^{(1)}, \ldots, \ell_{ji}^{(X_{ij})}$, where $\ell_{ji}^{(k)} = -C_{ij}^{(X_{ij}+1-k)}$.

3.) If arc (i,j) of the original network is saturated, draw a backward arc from node j to node i in the length sequence network. Label this backward arc with the

length sequence $\ell_{ji}^{(1)}, \ldots, \ell_{ji}^{(M_{ij})}$, where

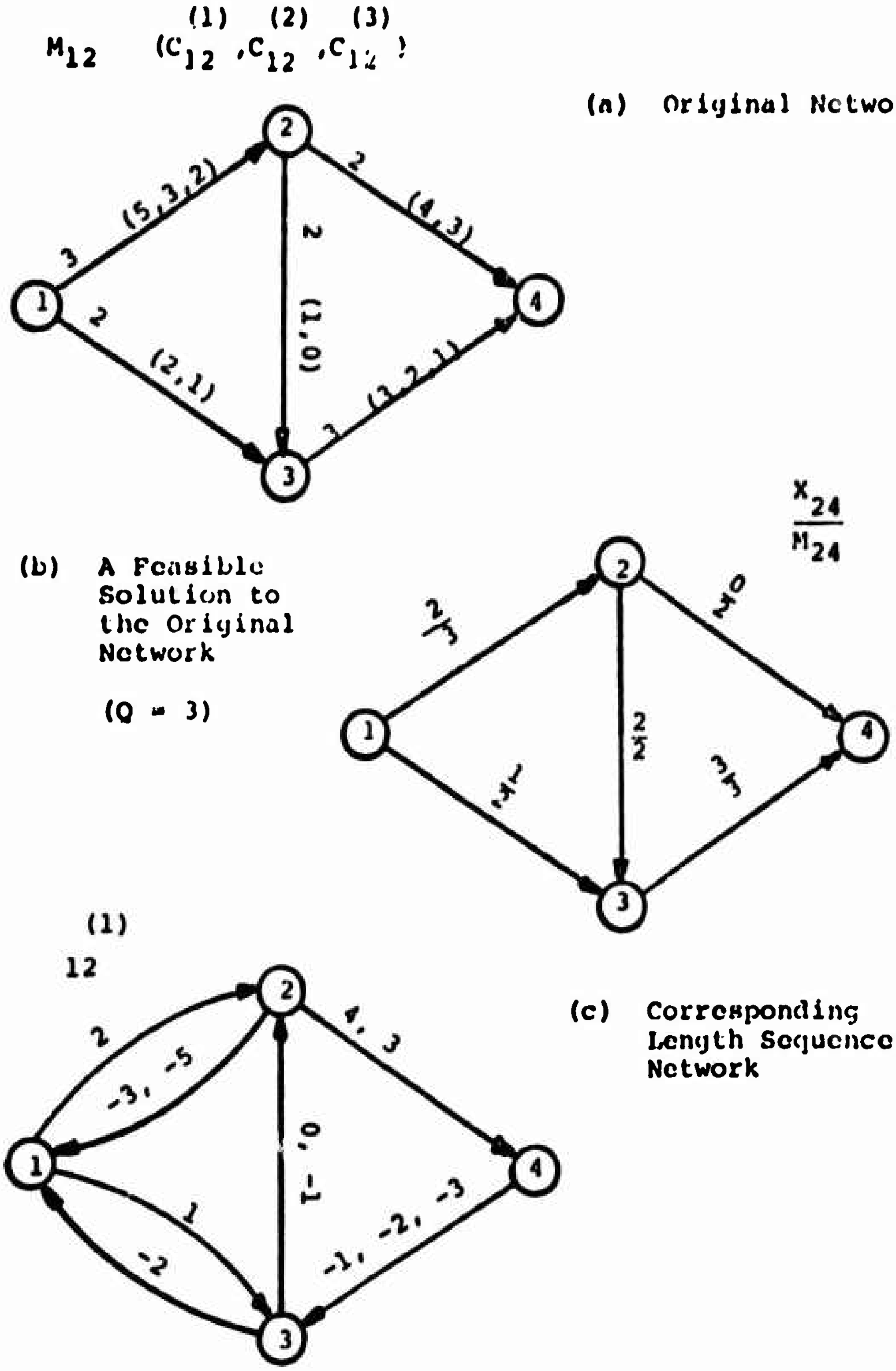
$$\ell_{ji}^{(k)} = -c_{ij}^{(M_{ij}+1-k)} .$$

Let the pair of "forward" and "backward" arcs corresponding to the same original network arc be called a complement. The number of complements in a length sequence network is equal to the number of arcs in the corresponding original network which are neither empty nor saturated.

Figure 1 illustrates how a length sequence network is constructed from and corresponds to a feasible solution of a simple network flow problem having $Q = 3$. Note the two complements $\{(1,2),(2,1)\}$ and $\{(1,3),(3,1)\}$ in the length sequence network corresponding to the two arcs (1,2) and (1,3) in the original network which are neither empty nor saturated.

For a forward arc (i,j) in the length sequence network, the sequence of length numbers corresponds to the sequence of incremental costs incurred with sending additional units of flow, in excess of X_{ij}, along the unsaturated arc (i,j) of the original network. Similarly, for a backward arc (j,i), the sequence of length numbers corresponds to the sequence of incremental costs incurred with "unsending," or removing, existing units of flow from the non-empty arc (i,j) of the original network. Due to the concave nature of the cost functions involved, the sequence of length numbers associated with each arc in the length sequence network is non-increasing.

Figure 1. Construction of a Length Sequence Network from a Feasible Solution of a Given Network Problem.



E. DEFINITIONS OF CYCLES

With respect to a length sequence network, a simple cycle will be defined as a series of adjacent and consistently directed arcs which:

(1) begins and ends at the same node,

(2) passes through at least two other nodes,

(3) does not have any arc in the series which is traversed more than once, and

(4) does not include both members of any complement.

If each arc of a simple cycle in a length sequence network has at least two elements in its length sequence, then it is possible to expand the simple cycle into a double cycle by identically and completely repeating the simple cycle. Similarly, if each arc of a simple cycle in a length sequence network has at least three elements in its length sequence, then it is possible to expand the simple cycle into a triple cycle by identically and completely repeating the simple cycle twice. A multiple cycle will be defined as any double cycle, triple cycle, et cetera.

A compound cycle will be defined to be a combination of two or more simple/multiple cycles such that:

(1) each simple/multiple cycle in the combination has at least one arc in common with another simple/multiple cycle in the combination,

(2) each arc common to two or more simple/multiple cycles has a sufficient number of elements in its length

sequence to accommodate all of the simple/multiple cycles to which it is common, and

(3) not more than one arc of any complement is included in the compound cycle.

The term cycle will hereafter be used to represent any of the specific cycles defined above. Figure 2 illustrates examples of cycles.

$\ell_{21}^{(1)}, \ell_{21}^{(2)}, \ell_{21}^{(3)}$

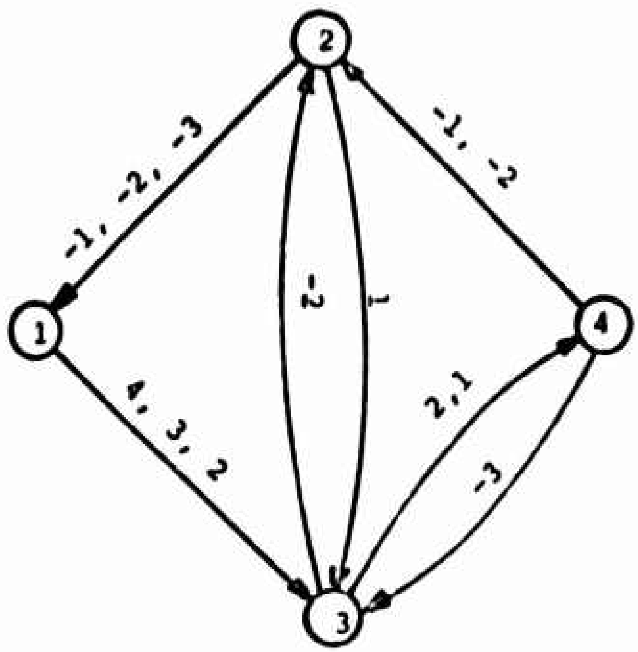


Figure 2. Examples of Cycles.

Simple Cycles: 1-3-4-2-1; 1-3-2-1; 2-3-4-2

Double Cycle: 1-3-4-2-1-3-4-2-1

Triple Cycle: None exist

Compound Cycles: 1-3-4-2-1-3-2-1; 2-1-3-4-3-4-2; 1-3-4-2-1-3-4-2-1-3-2-1

Note that 2-3-4-2-1-3-4-2-1-3-2 is not a legitimate cycle since it uses both members of the complement $\{(2,3),(3,2)\}$.

III. SOLUTION PROCEDURE

A. PREVIEW

The algorithm to be presented begins by determining any feasible solution to the given network problem. A length sequence network can then be constructed as described above. The identification of cycles in the length sequence network which have negative total length results in a recirculation of flow in the original network at a reduced total cost. This recirculation of flow results in a change of the existing feasible solution. A new length sequence is then created. The identification of cycles having negative length in this new length sequence network triggers another iteration. The algorithm terminates when no cycles having negative length can be identified in the length sequence network corresponding to the existing feasible solution of the original network. The existing feasible solution at the time of termination is an optimal solution to the problem.

The algorithm itself does not identify cycles in the length sequence network which have negative total lengths. Due to the large number of nodes and arcs which may exist in a length sequence network, identification of such cycles may be a difficult task. In order to terminate the algorithm, all possible cycles in the length sequence network must be checked to ensure that the existing feasible solution to the problem is an optimal solution.

B. THE ALGORITHM

1. Find any feasible solution to the problem and send flow across the network accordingly. Use (4) to determine the total cost associated with this initial feasible solution.

2. Based on the existing feasible flow, construct a length sequence network using the steps described in Section IID.

3. Try to find a cycle in the length sequence network which has a negative length. In attempting to identify such a cycle, two important conditions must be met:

a. If a particular arc in the length sequence network is traversed k times, then the first k elements in that arc's length sequence must be used in computing the cycle's total length.

b. Not more than one arc of any complement may be traversed.

4a. If a cycle of negative length can be identified, reallocate flow in the original network as follows:

(1) If forward arc (i,j) of the length sequence network is traversed k times by the cycle of negative length, increase the flow through arc (i,j) of the original network by k units.

(2) If backward arc (j,i) of the length sequence network is traversed k times by the cycle of negative length, decrease the flow through arc (i,j) of the original network by k units.

Also, determine the total cost associated with this new feasible flow. This will be equal to the cost associated with the previous feasible flow plus the length of the cycle which has just been identified.

Return to step 2.

4b. If a cycle of negative length cannot be found, then terminate the algorithm. The existing feasible flow, as well as the associated total cost, is optimal. If there exists a cycle of length zero, there is an alternate optimal solution to the problem obtainable by carrying out step 4a.

IV. EXAMPLE PROBLEM

Consider the network shown in Figure 3. The numerical quantities associated with each arc (i,j) are:

$$M_{ij} \quad (C_{ij}^{(1)}, C_{ij}^{(2)}, \ldots, C_{ij}^{(M_{ij})}).$$

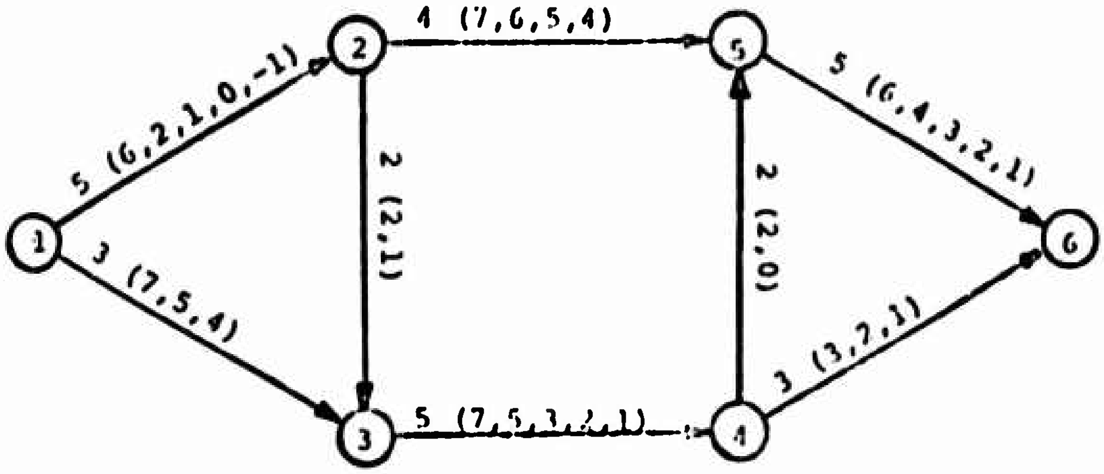


Figure 3. Example Network.

The alogrithm presented above will be used to solve this problem for a specified $Q = 4$.

Step 1.) Figure 4 shows an arbitrary initial feasible solution. The total cost associated with this initial feasible solution is: $Z = \$55$.

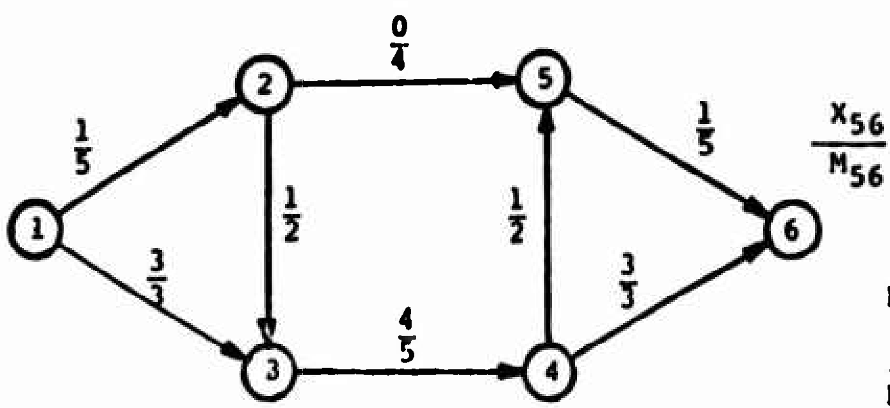


Figure 4.

Initial Feasible Solution

Step 2.) Figure 5 shows the length sequence network corresponding to this initial feasible flow.

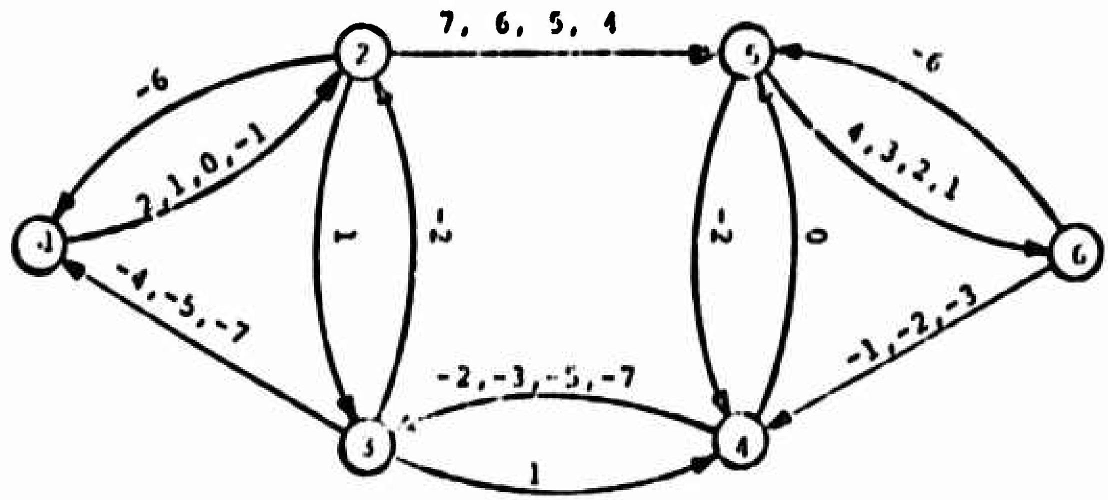


Figure 5. Initial Length Sequence Network.

Step 3.) The simple cycle 2-5-6-4-3-1-2 has a total length equal to $\ell_{25}^{(1)} + \ell_{56}^{(1)} + \ell_{64}^{(1)} + \ell_{43}^{(1)} + \ell_{31}^{(1)} + \ell_{12}^{(1)} = +6.$ Two complete cycles via 2-5-6-4-3-1-2 have a length of $+6 + 0 = +6$. Three complete cycles via 2-5-6-4-3-1-2 have a length of $+6 + 0 - 8 = -2$. Therefore, a cycle of negative length has been found.

Step 4a.) Forward arcs (1,2), (2,5), and (5,6) of the length sequence network are each traversed three times by the cycle of negative length which has just been identified. Therefore, the flow through arcs (1,2), (2,5), and (5,6) in the original network is increased by three units. Backward arcs (6,4), (4,3), and (3,1) of the length sequence network are each traversed three times by the cycle of negative length which has been identified. Therefore, the

flow through arcs (4,6), (3,4), and (1,3) in the original network is decreased by three units. Figure 6 shows the resulting reallocation of flow.

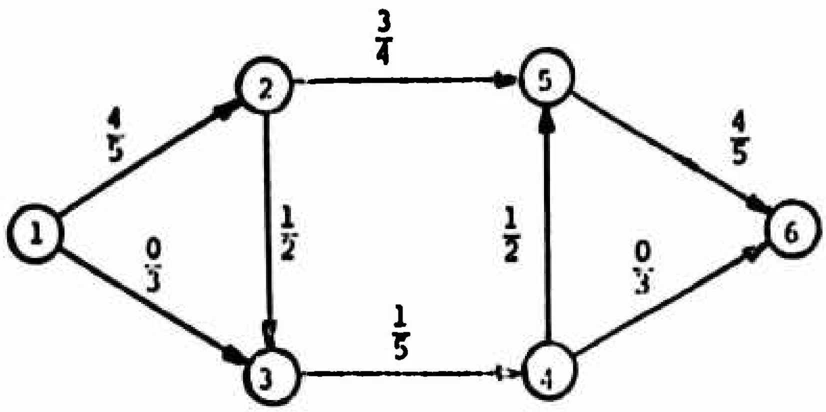


Figure 6. First Reallocation of Flow.

The total cost associated with this new feasible solution is: $Z = 55 - 2 = \$53$.

Step 2.) Figure 7 shows the length sequence network corresponding to this new feasible flow.

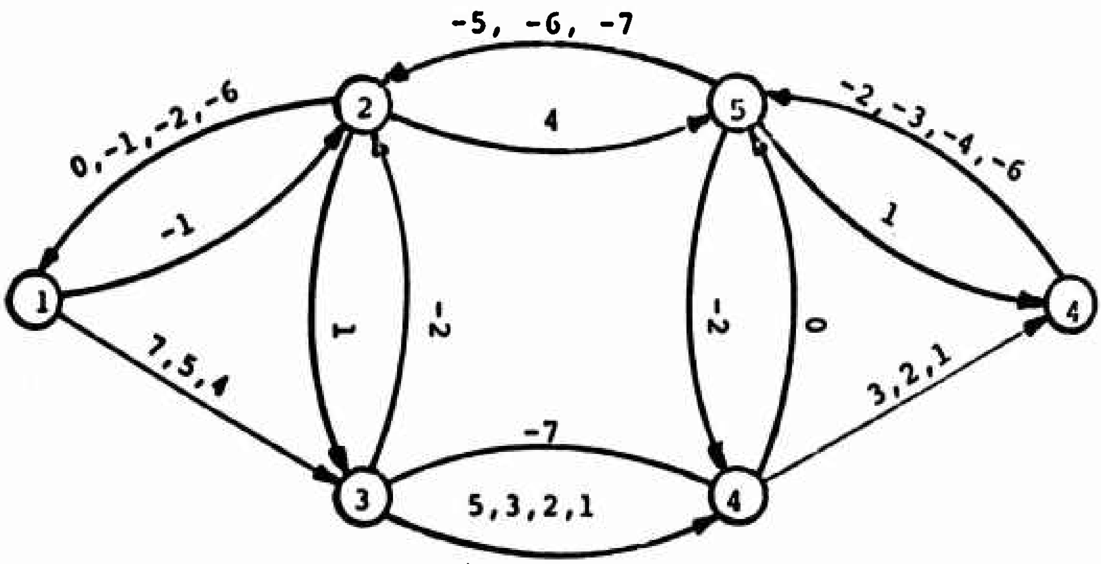


Figure 7. Second Length Sequence Network.

Step 3.) The simple cycle 2-5-4-3-2 in this second length sequence network has a total length equal to

$\ell_{25}^{(1)} + \ell_{54}^{(1)} + \ell_{43}^{(1)} + \ell_{32}^{(1)} = -7$. Therefore, a cycle of negative length has been found in this second length sequence network. Unfortunately, this cycle cannot be repeated.

Step 4a.) Since the forward arc (2,5) of the length sequence network is traversed one time by the recently discovered cycle of negative length, the flow through arc (2,5) in the original network is increased by one unit. Similarly, since the backward arcs (5,4), (4,3), and (3,2) of the length sequence network are each traversed one time by the recently discovered cycle of negative length, the flow through arcs (4,5), (3,4), and (2,3) in the original network is decreased by one unit. Figure 8 shows the resulting reallocation of flow.

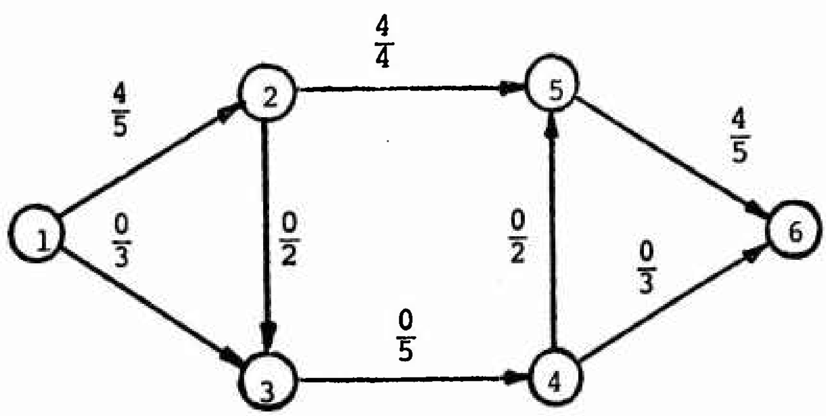


Figure 8. Second Reallocation of Flow.

The total cost associated with this new feasible solution is: Z = 53 - 7 = $46.

Step 2.) Figure 9 shows the length sequence network corresponding to this new feasible flow.

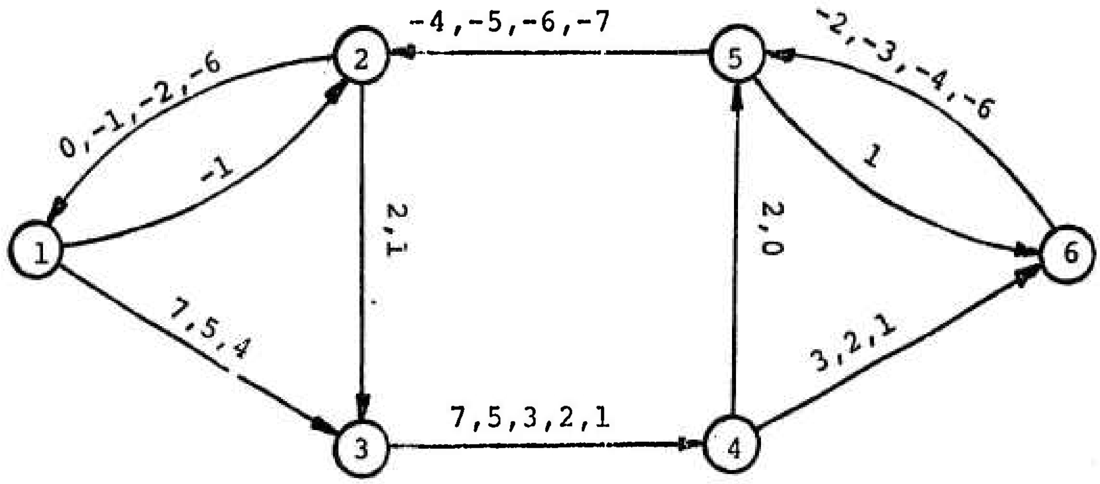


Figure 9. Third Length Sequence Network.

Step 3.) There are no cycles of negative length in this third length sequence network.

Step 4b.) The existing feasible flow, as illustrated in Figure 8, is optimal. The associated optimal cost of sending this flow through the original network is $46.

V. EXTENSIONS

A. SPECIAL CASE: $Q \leq$ MINIMUM M_{ij} VALUE

Consider a problem where the specified amount of flow Q to be sent through the network is less than or equal to the M_{ij} value of every arc in the network. Essentially, the problem presented by Zangwill [8] falls under this category, since he did not even consider the existence of upper bounds on arc flow capacities.

Such a problem can be solved by using the algorithm presented in this paper. However, an easier solution procedure to this problem exists. A minimum-cost chain from the source to the sink can be identified by employing any shortest route algorithm, such as those discussed by Dreyfus [4].

Let d_{ij} represent the length of the directed arc from node i to node j in a shortest route algorithm, such as the one developed by Dijkstra [3]. According to Dreyfus, Dijkstra's shortest route algorithm is computationally the most efficient.

For the above problem, let

$$d_{ij} = \begin{cases} C_{ij}(Q), & \text{if arc } (i,j) \text{ exists in the network} \\ \infty, & \text{if arc } (i,j) \text{ does not exist in the network} \end{cases} \tag{7}$$

To use Dijkstra's algorithm, all d_{ij} values must be greater than or equal to zero. Since the cost functions

are assumed to be non-negative over the entire range of their respective feasible arc flows, this condition is met.

Also, let $V_i^{(k)}$ denote the label value of node i after the kth iteration of Dijkstra's algorithm has been completed. Dijkstra's algorithm proceeds as follows:

Step 0.) Set $V_1^{(0)} = 0$, $V_i^{(0)} = \infty$, $i = 2,3,\ldots,n$.

Declare $V_1 = V_1^{(0)} = 0$ as the permanent label value of node 1.

Step 1.) Compute $V_{j\neq 1}^{(1)} = \min\left\{V_j^{(0)},\ V_1+d_{1j}\right\}$. Then find $V_p^{(1)} = \min V_j^{(1)}$ and specify $V_p = V_p^{(1)}$ as the permanent label value of node p.

Step 2.) Compute $V_{j=1,p}^{(2)} = \min\left\{V_j^{(1)},\ V_p+d_{pj}\right\}$. Then find $V_q^{(2)} = \min V_j^{(2)}$ and specify $V_q = V_q^{(2)}$ as the permanent label value of node q.

Step 3.) Continue the implied iterative process until node n has a permanent label value.

At most, n-1 iterations will be required to label node n. The minimum-cost chain will consist of the arcs (1,p), (p,q), (q,r),..., (m,n). Send Q units of flow along this chain.

B. NON-ZERO LOWER BOUNDS

The algorithm presented in this paper can be modified to handle problems in which non-zero lower bounds are associated with the arcs of the network. Let L_{ij} be the lower bound on flow capacity for an arc (i,j). Then for X_{ij} to be a feasible arc flow,

$$L_{ij} \le X_{ij} \le M_{ij} \tag{8}$$

for each arc (i,j) in the network.

Constraint (6) in the cost-minimization problem is replaced by (8). Once again, the algorithm begins by finding any flow which satisfies the constraints. The only change in the algorithm is that a length sequence network is constructed as follows:

1.) If arc (i,j) of the original network is at its lower bound, draw a forward arc from node i to node j in the length sequence network. Label this forward arc with the length sequence $\ell_{ij}^{(1)}, \ldots, \ell_{ij}^{(M_{ij}-L_{ij})}$, where $\ell_{ij}^{(k)} = C_{ij}^{(L_{ij}+k)}$.

2.) If arc (i,j) of the original network is neither at its lower bound nor saturated, draw a forward arc from node i to node j and draw a backward arc from node j to node i in the length sequence network. Label the forward arc with the length sequence $\ell_{ij}^{(1)}, \ldots, \ell_{ij}^{(M_{ij}-X_{ij})}$, where $\ell_{ij}^{(k)} = C_{ij}^{(X_{ij}+k)}$. Label the backward arc with the length sequence $\ell_{ji}^{(1)}, \ldots, \ell_{ji}^{(X_{ij}-L_{ij})}$, where $\ell_{ji}^{(k)} = -C_{ij}^{(X_{ij}+1-k)}$.

3.) If arc (i,j) of the original network is saturated, draw a backward arc from node j to node i in the length sequence network. Label this backward arc with the length sequence $\ell_{ji}^{(1)}, \ldots, \ell_{ji}^{(M_{ij}-L_{ij})}$, where $\ell_{ji}^{(k)} = -C_{ij}^{(M_{ij}+1-k)}$.

VI. AREAS FOR FURTHER STUDY

A. IDENTIFICATION OF NEGATIVE CYCLES

The only drawback with the algorithm presented in this paper is that it includes no procedure for identifying cycles of negative length in the length sequence network. Since the identification of such cycles generates the iterative process of the algorithm, the discovery of an organized procedure which identifies such cycles would be most useful.

Yen [7] has devised an efficient shortest route algorithm which makes forward and backward scanning passes on a matrix composed of d_{ij} values as defined above. As a by-product of his algorithm, simple cycles of negative length can easily be identified. The attempts by the author to identify multiple and compound cycles by modifying Yen's matrix were not successful.

B. GENERAL COST FUNCTIONS

The algorithm presented in this paper can also be used to solve problems in which the cost functions are linear or convex. Due to the non-decreasing nature of the related cost increments and length sequences, it would only be necessary to find simple cycles of negative length. The restriction against using both members of a complement could be waived.

It is the author's contention that the algorithm could be modified to solve problems in which the cost functions are neither convex nor concave. Further research in this area might prove to be very interesting.

VII. SUMMARY

A procedure has been developed and presented for optimally sending any specified amount of flow through a network in which each arc has an upper bound on capacity and a concave cost function.

The solution technique involves the construction of a length sequence network corresponding to a feasible solution of the problem. The identification of negative cycles in the length sequence network results in a recirculation of flow in the original network and yields a reduction in the overall cost. Recirculation of flow in the original network creates another length sequence network. The iterative process continues until no further negative cycles can be identified. When this occurs, the existing feasible flow in the original network is optimal.

Inputs needed to solve the problem are the upper bound on capacity and the cost function associated with each arc in the network.

Modifications to the solution procedure are presented so that network problems having the additional characteristic of non-zero lower bounds may be solved. A very simple substitute method is also presented to deal with a special case of the problem: $Q \leq$ minimum M_{ij} value in the network.

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